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Paths to Stability in Two-sided Matching with Uncertainty

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Summary

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Keywords: Consistent Outcomes, One-to-One Uncertainty, Many-to-One Uncertainty, Paths to Stability, Two-Sided Matching

JEL Classification: C62, C78, D71, D83

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Abstract

We consider one-to-one matching problems under two modalities of uncertainty that differ in the way types are assigned to agents. Individuals have preferences over the possible types of the agents from the opposite market side and initially know the 'name' but not the 'type' of the other players. Learning occurs via matching and using Bayes' rule. We introduce the notion of a stable and consistent outcome, and show how the interaction between blocking and learning behavior shapes the existence of paths to stability in each of the uncertainty environments. Existence of stable and consistent outcomes then follows as a side result.

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A second marriage is the triumph of hope over experience.
Samuel Johnson

1 Introduction

Since the seminal contribution of Gale and Shapley (1962) economists have predominantly analyzed centralized mechanisms to derive equilibrium outcomes in two-sided markets. The question whether such outcomes can be reached in a decentralized manner has taken a second stage although, one may argue, the decentralized markets outnumber those with a centralized mechanism in place. To the best of our knowledge the only authors to address this question are Roth and Vande Vate (1990), Diamantoudi et al. (2004), and Chen et al. (2011). In all of these works, however, it is assumed that players on both sides of the market have complete information about the type of the agents on the other market side.

In the present paper we re-visit the question on whether an equilibrium outcome in the standard one-to-one two-sided market model can be reached in a decentralized manner when we assume away complete information. We discuss instead two worlds of uncertainty that differ in terms of the process that assigns types to market players. Here market participants have preferences over the types of the agents with whom they can be matched but not over their identities. In our analysis information requirements are kept to the minimum, that is, players only know their own type and type is independent of individual preferences. Thus, two agents of the same type may have different preferences. This constitutes also a crucial difference between our work and Liu at al. (2012), apart from the fact that we focus on the existence of paths leading to stable and consistent outcomes (consisting of a matching and a corresponding system of beliefs) in this framework.

More precisely, we show that when the number of types equals the number of agents and types are assigned as random independent draws from the set of types without replacement (one-to-one uncertainty), then any stable matching under complete information is part of a stable and consistent outcome of the corresponding matching problem with one-to-one uncertainty (Section 3.1). Consistency is shown in this case by the existence of a path containing a multiple of three steps of a particular form. On the other hand, when the number of types is at most equal to the number of corresponding agents and types are assigned to agents as random independent draws from the set of types with replacement (many-to-one uncertainty), we show that, starting from an arbitrary self-consistent outcome, there exists a path to a stable and consistent outcome for any matching problem with many-to-one uncertainty (Section 3.2). The construction of a path in this case is shaped by the interaction between blocking and learning behavior and uses, for some of its parts, Roth and Vande Vate's (1990) algorithm for reaching a stable matching in environments with complete information.

2 Setup

We consider two finite sets M and W of agents, called "men" and "women", respectively. Agents can be of different types. We denote the finite set of all possible male types by Θ^M and the finite set of all possible female types by Θ^W. The functions $\theta^M : M \to \Theta^M$ and $\theta^W : W \to \Theta^W$ assign a type to each man and woman, respectively. Male's preferences are defined over all possible female types and the possibility of remaining single, and are assumed to be complete, transitive and antisymmetric. Correspondingly, female's strict preferences are defined over all possible male types and the possibility of remaining single. A profile of such preferences is denoted by $\succeq = (\succeq_i)_{i \in M \cup W}$. When the assignment of types is known, agents can use their preferences over types to derive preferences over individuals on the other side of the market. Notice, however, that strict preferences over types do not imply strict preferences over agents as some agents can be of the same type.

Initially individuals know their own type (and thus, the "type" of the possibility of remaining single) and only the 'name' of all other individuals from the opposite market side but not their types. We assume, instead, that all agents have a common prior about the distribution of types among players. Notice that it is not necessary for an agent to have any information about the distribution of types among agents on their own side of the market. Thus, priors can be women and men specific, and we denote them by π^W and π^M, respectively. A *one-to-one matching problem with uncertainty* is a collection of data consisting of two finite sets of agents, the corresponding finite sets of types, assignment functions, priors, as well as a preference profile over types.

In the process of matching, agents learn the type of their partners and can use Bayes' rule to update their priors on the type of agents on the other side of the market with whom they have not been matched. Therefore, we define an *outcome of the matching problem under uncertainty* as a pair (μ, α) consisting of a *matching function* μ and a *system of beliefs* α. The matching function $\mu : M \cup W \to M \cup W$ is such that $\mu(i) \in W \cup \{\emptyset\}$, $\mu(j) \in M \cup \{\emptyset\}$, and $\mu^2(k) = k$ hold for all $i \in M$, all $j \in W$, and all $k \in M \cup W$. The interpretation of $\mu(k) = \emptyset$ for some $k \in M \cup W$ is that the corresponding agent is single under μ. The system of beliefs α contains all agents' beliefs about the type of each agent on the opposite side of the market. In particular, we use the notation $\alpha_i(j, t)$ to denote the belief agent i holds about j being of type t.

Using the system of beliefs, we can define a *blocking pair* for an outcome (μ, α). A pair of agents (m, w) with $m \in M$ and $w \in W$ is blocking the outcome (μ, α) if there are a female type $t_1 \in \Theta^W$ and a male type $t_2 \in \Theta^M$ such that the following two conditions hold:

(1) $t_1 \succ_m \theta^W(\mu(m))$ and $t_2 \succ_w \theta^M(\mu(w))$;

(2) $\alpha_m(w, t_1) > 0$ and $\alpha_w(m, t_2) > 0$.

Thus we require that each member of a blocking pair assigns positive probability to the fact that the other pair member is of a type ranked higher than the type of his or her current match. Note that the possibility of an agent blocking unilaterally the matching μ is also captured in the above formulation.

Certainly, the beliefs that an agent holds evolve with the search of an optimal partner, thus they cannot be just any believes but should be consistent with the individual agent's history. We call a system of beliefs α *consistent with respect to a matching* μ (denoted by $\alpha_{|\mu}$) if the following conditions are met:

(1) for all $m \in M$ with $\mu(m) \neq \emptyset$, $\alpha_m(\mu(m), \theta^W(\mu(m))) = 1$ and $\alpha_m(\mu(m), t) = 0$ for all $t \in \Theta^W \setminus \{\theta^W(\mu(m))\}$, and $\alpha_m(w, t) = Prob(\theta^W(w) = t \mid \theta^W(\mu(m)))$ for all $w \in W \setminus \{\mu(m)\}$ and all $t \in \Theta^W$.

(2) for all $w \in W$ with $\mu(w) \neq \emptyset$, $\alpha_w(\mu(w), \theta^M(\mu(w))) = 1$ and $\alpha_w(\mu(w), t) = 0$ for all $t \in \Theta^M \setminus \{\theta^M(\mu(w))\}$, and $\alpha_w(m, t) = Prob(\theta^M(m) = t \mid \theta^M(\mu(w)))$ for all $m \in M \setminus \{\mu(w)\}$ and all $t \in \Theta^M$.

(3) for all $m \in M$ with $\mu(m) = \emptyset$, $\alpha_m(w, t) = \pi^W(t)$ for all $w \in W$ and all $t \in \Theta^W$.

(4) for all $w \in W$ with $\mu(w) = \emptyset$, $\alpha_w(m,t) = \pi^M(t)$ for all $m \in M$ and all $t \in \Theta^M$.

Here the consistency of the system of beliefs with respect to a matching requires first, that each agent knows the type of his or her partner in this matching; and second, that agents' beliefs about the type of all other agents with whom they are not matched are updated using Bayes' rule. Notice in addition that agents staying single in the matching do not update their beliefs, i.e., their beliefs about the type of all agents on the opposite market side are given by the corresponding common priors. The outcome $\left(\mu, \alpha_{|\mu}\right)$ is called *self-consistent.*

Next, we define the consistency of an outcome with respect to a given history of matchings. We will consider an outcome (μ, α) to be *consistent with respect to a self-consistent initial outcome* $(\mu_0, \alpha_{|\mu_0})$ if there is a sequence of outcomes $\left(\mu_1, \alpha_{|\mu_1}\right), \ldots, \left(\mu_k, \alpha_{|\mu_1,\ldots,\mu_k}\right)$ with $\left(\mu_1, \alpha_{|\mu_1}\right) = (\mu_0, \alpha_{|\mu_0})$ and $\left(\mu_k, \alpha_{|\mu_1,\ldots,\mu_k}\right) = (\mu, \alpha)$ such that for all $\ell = 1, \ldots, k-1$:

(1) there is a blocking pair (m_ℓ, w_ℓ) for $\left(\mu_\ell, \alpha_{|\mu_1,\ldots,\mu_\ell}\right)$ such that $\mu_{\ell+1}$ is obtained from μ_ℓ by satisfying (m_ℓ, w_ℓ);

(2) there is a consistent Bayesian updating of beliefs $\alpha_{|\mu_1,\ldots,\mu_{\ell+1}}$ such that for all $\ell = 1, \ldots, k-1$:

(2.1) $\alpha_{m_\ell}\left(w_\ell, \theta^W(w_\ell)\right)_{|\mu_1,\ldots,\mu_{\ell+1}} = \alpha_{w_\ell}\left(m_\ell, \theta^M(m_\ell)\right)_{|\mu_1,\ldots,\mu_{\ell+1}} = 1$;

(2.2) $\alpha_{m_\ell}\left(w_\ell, t\right))_{|\mu_1,\ldots,\mu_{\ell+1}} = 0$ for all $t \in \Theta^W \backslash \{\theta^W(w_\ell)\}$ and $\alpha_{w_\ell}\left(m_\ell, t\right))_{|\mu_1,\ldots,\mu_{\ell+1}} = 0$ for all $t \in \Theta^M \setminus \{\theta^M(m_\ell)\}$;

(2.3) $\alpha_{m_\ell}\left(w, t\right)_{|\mu_1,\ldots,\mu_{\ell+1}} = Prob(\theta^W(w) = t \mid \theta^W(w_\ell), \alpha_{|\mu_1,\ldots,\mu_\ell})$ for all $w \in W \setminus \{w_\ell\}$ and all $t \in \Theta^W$, and $\alpha_{w_\ell}\left(m, t\right)_{|\mu_1,\ldots,\mu_{\ell+1}} = Prob(\theta^M(m) = t \mid \theta^M(m_\ell), \alpha_{|\mu_1,\ldots,\mu_\ell})$ for all $m \in M \setminus \{m_\ell\}$ and all $t \in \Theta^M$;

(2.4) $\alpha_m(w,t)_{|\mu_1,\ldots,\mu_{\ell+1}} = \alpha_m(w,t)_{|\mu_1,\ldots,\mu_\ell}$ for all $m \in M \setminus \{m_\ell\}$ and all $t \in \Theta^W$, and $\alpha_w(m,t)_{|\mu_1,\ldots,\mu_{\ell+1}} = \alpha_w(m,t)_{|\mu_1,\ldots,\mu_\ell}$ for all $w \in W \setminus \{w_\ell\}$ and all $t \in \Theta^M$.

Clearly, condition (1) above defines a 'legitimate' path of search for an optimal partner. We take an outcome to be consistent with respect to an initial self-consistent outcome if it can be derived from it by satisfying blocking pairs. Condition (2), on the other hand, describes a sound 'learning process', i.e., the updating of beliefs along the path of blocked matchings. We require here that all agents who are matched to each other know their true type; these agents use Bayesian updating to re-calculate the probability with which any other agent on the opposite side of the market is of any given type; and last, agents who do not participate in a blocking pair do not update their beliefs as they do not gain any additional information.

Using the above definitions, we can define an outcome (μ, α) to be *consistent* if there exists an initial self-consistent outcome $(\mu_0, \alpha_{|\mu_0})$ with respect to which it is consistent. An outcome (μ, α) is *stable* if there are no blocking pairs for it. In what follows we will focus on outcomes which are both stable and consistent.

3 World of uncertainty

We will discuss two different mechanisms that map agents to types. In the first one learning the type of one agent will be informative about the probability with which all other agents on the opposite side of the market are of a particular type, while for the second mechanism this will not be the case. Hence in the second case learning will be slower. In each of these cases we will discuss the relation between the set of stable and consistent outcomes under uncertainty and the set of stable outcomes under complete information. We will also ask the question whether there is a path reaching a stable and consistent outcome starting from any initial self-consistent outcome.

To answer the former question, we need to recall here the standard definition of a matching problem, how it is related to a matching problem under uncertainty, and what constitutes a stable outcome under complete information. A one-to-one matching problem with complete information is a tuple $(M, W, \succeq')$, where M and W are the sets of men and women as defined above and $\succeq'$ denotes a preference profile that collects the preferences that men and women hold over their potential partners in a matching. Given a matching problem under uncertainty as defined above, we say that the matching problem with complete information $(M, W, \succeq')$ corresponds to it if the sets of agents coincide and the preference profiles are such that for all agents they induce the same ranking of potential partners. That is, for $m \in M$ and $w_i, w_j \in W$, $w_i \succeq'_m w_j$ if and only if $\theta^W(w_i) \succeq_m \theta^W(w_j)$, and similarly, for $w \in W$ and $m_i, m_j \in M$, $m_i \succeq'_w m_j$ if and only if $\theta^M(m_i) \succeq_w \theta^M(m_j)$. A matching μ is stable under complete information if there does not exist a pair (m, w) of agents such that $w \succ'_m \mu(m)$ and $m \succ'_w \mu(w)$.

3.1 One-to-one uncertainty

Consider a situation in which the number of male and female types equals the number of men and women, respectively, and types are assigned as random independent draws from the set of corresponding types without replacement, i.e., there is a one-to-one mapping between identities and types (θ^M and θ^W are bijections). Thus, the prior belief that each man holds about the type of any woman is given by $\pi^W(t) = \frac{1}{|W|}$ for all $t \in \Theta^W$, and $\pi^M(t) = \frac{1}{|M|}$ for all $t \in \Theta^M$ is the prior probability that any man is of any given type.

Here knowing the type of one partner is informative about what types other potential partners may be, and more importantly, the probability with which other potential partners are ranked higher than the current one. Moreover, as agents are endowed with strict preferences over types, it implies that their corresponding preferences over potential partners are also strict. We will refer to this case as *one-to-one uncertainty*.

The existence of stable and consistent outcomes in this case is a direct corollary of our first result.

Theorem 1 *Let a matching problem under one-to-one uncertainty be given and (μ, α) be an outcome of it. Then (μ, α) is stable and consistent if and only if μ is stable for the corresponding matching problem under complete information.*

Proof. First we show that any matching which is part of a stable and consistent outcome under one-to-one uncertainty is also a stable matching under complete information. Consider a stable and consistent outcome (μ, α) under one-to-one uncertainty, and suppose that μ is not stable for the corresponding matching problem $(M, W, \succeq')$ under complete information. Therefore, there exists a pair (m, w) of agents who are not matched to each other under μ and prefer to be matched to each other than to their current partners: $w \succ'_m \mu(m)$ and $m \succ'_w \mu(w)$. This implies that $\theta^W(w) \succ_m \theta^W(\mu(m))$ and $\theta^M(m) \succ_w \theta^M(\mu(w))$. Given the consistency of agents' beliefs and $\pi^M(t) > 0$ for all $t \in \Theta^M$ and $\pi^W(t) > 0$ for all $t \in \Theta^W$, it must be that both m and w hold strictly positive beliefs that the other agent is of their true type, i.e., $\alpha_w(m, \theta^M(m)) > 0$ and $\alpha_m(w, \theta^W(w)) > 0$. Therefore, by setting $t_1 = \theta^W(w)$ and $t_2 = \theta^M(m)$, (m, w) is a blocking pair for the outcome (μ, α) under one-to-one uncertainty, too. Thus, we have established a contradiction.

Let us now consider the matching problem $(M, W, \succeq')$ under complete information and let μ be a stable matching for this problem. We will show that there is a consistent outcome (μ, α) of the corresponding problem under one-to-one uncertainty which is also stable. Consider the initial self-consistent outcome $(\mu, \alpha_{|\mu})$. If there are no blocking pairs in $(\mu, \alpha_{|\mu})$, then we have shown what we need. Notice further that it is impossible for an agent to block $(\mu, \alpha_{|\mu})$ unilaterally as μ is stable under complete information and thus, individually rational. Suppose now that there is a pair (m, w) that blocks $(\mu, \alpha_{|\mu})$. That is, there are a female type $t_1 \in \Theta^W$ and a male type $t_2 \in \Theta^M$ such that (1) $t_1 \succ_m \theta^W(\mu(m))$ and $\alpha_m(w, t_1)_{|\mu} > 0$, and (2) $t_2 \succ_w \theta^M(\mu(w))$ and $\alpha_w(m, t_2)_{|\mu} > 0$. It follows then that we can construct the consistent outcome $(\mu_1, \alpha_{|\mu,\mu_1})$. This cannot be a stable outcome: since μ is stable, then either $\mu(m) \succ'_m w$ and thus, $\theta^W(\mu(m)) \succ_m \theta^W(w)$, or $\mu(w) \succ'_w m$, thus $\theta^M(\mu(w)) \succ_w \theta^M(m)$. Suppose it is m who forms a blocking pair $(m, \mu(m))$ with his partner in μ. By satisfying this blocking pair we can construct the consistent outcome $(\mu_2, \alpha_{|\mu,\mu_1,\mu_2})$. This consistent outcome cannot be stable either as w forms a blocking pair $(\mu(w), w)$ with her partner in μ, the reason being that μ is individually rational and preferences in both matching problems (with one-to-one uncertainty and with complete information) are strict. By satisfying this blocking pair we construct the consistent outcome $(\mu_3, \alpha_{|\mu,\mu_1,\mu_2,\mu_3})$, where by construction $\mu_3 = \mu$ and $\alpha_{|\mu,\mu_1,\mu_2,\mu_3} = \alpha_{|\mu,\mu_1}$.

Consider finally the consistent outcome $(\mu, \alpha_{|\mu,\mu_1,\mu_2,\mu})$. The pair (m, w) cannot block this matching because in the process of beliefs' updating m has learned the type of w and knows that he prefers to be with his partner in μ than to be with w. If there is no blocking pair, then this is a stable outcome and we have shown what we need. If there is a blocking pair, then this pair was also blocking the initial self-consistent outcome $(\mu, \alpha_{|\mu})$. Then, using the same logical steps as above, we can construct a path by satisfying the blocking pairs that will lead to a consistent outcome in a multiple of three steps that comprises of μ and a system of beliefs in which exactly four agents (two men and two women) use Bayes' rule to update their beliefs in a consistent manner. The process will continue in a multiple of three steps along the path until all agents who form blocking pairs in $(\mu, \alpha_{|\mu})$ have learned the type of their partners in the blocking pair. Since μ is stable under complete information, at least one of the partners in these blocking pairs will prefer her or his partner in μ to the one with whom they formed a blocking pair under one-to-one uncertainty. Thus, we can always go back to μ. Due to the finiteness of the sets M and W, this path will terminate in a finite number of steps with a stable and consistent outcome that contains μ. ■

Given the existence result of Gale and Shapley (1962) for stable outcomes in the standard one-to-one matching problem, it is easy to establish the non-emptiness of the set of stable and consistent outcome under one-to-one type of uncertainty as a corollary of the above result.

Corollary 1 *The set of stable and consistent outcomes for any matching problem under one-to-one uncertainty is non-empty.*

3.2 Many-to-one uncertainty

Consider a situation in which the number of types is at most equal to the number of the corresponding agents (men and women), and types are assigned to agents as random independent draws from the set of types with replacement (i.e., $\pi^W(t) = \frac{1}{|\Theta^W|}$ for all $t \in \Theta^W$ and $\pi^M(t) = \frac{1}{|\Theta^M|}$ for all $t \in \Theta^M$). Since the assignment functions θ^M and θ^W may not be one-to-one or onto and thus, many agents can be assigned the same type, we have that agents' preferences over potential partners can contain indifferences even though their preferences over types are strict. We refer to this case as *many-to-one uncertainty.*

Here knowing the type of one partner is not informative about the types of the other agents on the opposite market side, and more importantly, the probability with which (the type of) the other potential partners are ranked higher than (the type of) the current one. Consequently, agents will continue 'learning' by blocking any matching in which they do not have complete information unless they are matched to an agent of their most preferred type. This observation will be in the core of the proof of our next result.

Theorem 2 *Let a matching problem under many-to-one uncertainty be given and* $(\mu_0, \alpha_{|\mu_0})$ *be a self-consistent outcome of it. Then the matching problem has a stable outcome which is consistent with respect to* $(\mu_0, \alpha_{|\mu_0})$.

Proof. The proof will be constructive. Let us collect in the set $B(0)$ all agents who form blocking pairs for $(\mu_0, \alpha_{|\mu_0})$ such that the corresponding pair members know each other, and let $L(0)$ be the analogous set in which the members of a blocking pair do not know each other, i.e., there is a possibility of learning. We can then define the set $S(0) = \{\{m, w\} \subseteq (M \cup W) \setminus \{L(0) \cup B(0)\} : \mu_0(m) = w,\ \alpha_w(m_i, \theta^M(m_i)) < 1$ and $\alpha_m(w_j, \theta^W(w_j)) < 1$ for some $m_i, w_j \in B(0) \cup L(0)\}$ consisting of all married agents under μ_0 who will not form a blocking pair in any subsequent matching because the fact that they do not form a blocking pair for $(\mu_0, \alpha_{|\mu_0})$ with the possibility for learning with someone from $B(0)$ or $L(0)$ implies that these agents are matched to partners of their most preferred type. If there is no blocking pair at all for $(\mu_0, \alpha_{|\mu_0})$, we are done. Given the self-consistency of $(\mu_0, \alpha_{|\mu_0})$, we have $B(0) = \emptyset$. So, if there is a blocking pair for $(\mu_0, \alpha_{|\mu_0})$, then it must contain agents only from $L(0)$.

In this case we can construct a sequence of consistent outcomes $(\mu_0, \alpha_{|\mu_0})$,

$(\mu_1, \alpha_{|\mu_0,\mu_1})$, , . . ., $(\mu_k, \alpha_{|\mu_0,\mu_1,\ldots,\mu_k})$ along which individuals can learn the type of the agents on the opposite side of the market by forming blocking pairs only with such agents with whom they have not been matched before. Here k is the smallest integer for which $L(k) = \emptyset$, i.e., there is no possibility for learning. Consider the consistent outcome $(\mu_k, \alpha_{|\mu_0,\mu_1,\ldots,\mu_k})$ and note that if $B(k) = \emptyset$, then we are done. If, however, $B(k) \neq \emptyset$, then all men (women) in $B(k)$ must know the type of all women (men) in $B(k)$, otherwise they could form a blocking pair with learning in contradiction to $L(k) = \emptyset$.

If $B(k) = (M \cup W) \setminus S(k)$, then we can use the algorithm of Roth and Vande Vate (1990) to construct a stable matching of agents in $B(k)$. This will lead to a stable and consistent outcome in which the agents in $S(k)$ will be matched according to μ_k, agents in $B(k)$ will be matched according to Roth and Vande Vate's (1990) algorithm for reaching a stable matching, and the beliefs along the path will equal $\alpha_{|\mu_0,\mu_1,\ldots,\mu_k}$, i.e., there will be no further updating of beliefs because only agents who know each other's type will be matched. In case $B(k) \neq (M \cup W) \setminus S(k)$, we can pick up at random one of w_k's most preferred partners in $B(k)$, say m_k, and construct the consistent outcome $(\mu_{k+1}, \alpha_{|\mu_0,\mu_1,\ldots,\mu_{k+1}})$ by satisfying the blocking pair (m_k, w_k) and setting $\alpha_{|\mu_0,\mu_1,\ldots,\mu_{k+1}} = \alpha_{|\mu_0,\mu_1,\ldots,\mu_k}$. Set $A(k+1) = \{m_k, w_k\}$ to be the set of satisfied blocking pairs where agents knew each other's type prior to this matching.

If $L(k+1) = \emptyset$ and $B(k+1) = \emptyset$, then we are done. If $L(k+1) \neq \emptyset$, however, then construct μ_{k+2} by satisfying a blocking pair in $L(k+1)$ and update the beliefs in a consistent manner. Set $A(k+2) = \emptyset$. Notice that $L(q) = \emptyset$ in some finite steps q due to the finiteness of the sets M and W, i.e., men and women will eventually learn the types of all agents on the opposite side of the market. And if $L(k+1) = \emptyset$, but $B(k+1) \neq \emptyset$, then it must be that all agents in $B(k+1)$ and $A(k+1)$ know each other's type otherwise they would have formed a blocking pair with learning in μ_k or μ_{k+1}. Also notice that $w_k \notin B(k+1)$ because m_k is one of w_k's most preferred partners and she cannot form any new blocking pairs in μ_{k+1} that she could not form in μ_k. Then pick a blocking pair at random from the set $B(k+1)$, say (w_{k+1}, m_{k+1}) and form the matching μ_{k+2} by satisfying this blocking pair. Let $\alpha_{|\mu_0,\mu_1,\ldots,\mu_{k+2}} = \alpha_{|\mu_0,\mu_1,\ldots,\mu_{k+1}} = \alpha_{|\mu_0,\mu_1,\ldots,\mu_k}$. Set $A(k+2) = A(k+1) \cup \{m_{k+1}, w_{k+1}\}$ and note that $A(k+1) \subseteq A(k+2)$.

Thus, if there is no subsequent step r with $L(r) \neq \emptyset$ (i.e., there are no possibilities for learning any more), we can adopt Roth and Vande Vate's (1990) algorithm to construct an increasing sequence of sets that contain no blocking pairs until a stable matching is found. This is possible because, as argued above, in any outcome in which all agents who form a blocking pair know each other's type, they must also know the type of any other agent on the opposite side of the market who forms a blocking pair; and they either know the type of the agents in the set $A(r)$ with whom they do not form a blocking pair, or those agents in $A(r)$ whose type they do not know are also in $S(r)$ and thus they are matched to partners of their most preferred type. Since only blocking pairs with no learning are satisfied along the path following μ_k and reaching a stable matching, we construct a stable and consistent outcome that consists of the stable matching just obtained and the system of beliefs $\alpha_{|\mu_0,\mu_1,\ldots,\mu_k}$. ■

An immediate corollary of Theorem 2 is the existence of a stable and consistent outcome in this set up.

Corollary 2 *The set of stable and consistent outcomes for any matching problem under many-to-one uncertainty is non-empty.*

It is also straightforward to show the following relation to the set of stable matchings in the corresponding complete information problem.

Proposition 1 *Let a matching problem under many-to-one uncertainty be given and* (μ, α) *be a stable and consistent outcome of it. Then* μ *is stable for the corresponding matching problem with complete information.*

The proof of Proposition 1 is analogous to the first part of the proof of Theorem 1.

4 Conclusion

In this work we embed the standard one-to-one matching problem in an environment of uncertainty. We show that with very little information requirements we can replicate standard results from the theory under complete information. Thus, one may argue assuming complete information in the first place has not been a limitation. On the other hand, developing a methodology for the analysis of two-sided matching problems under uncertainty opens the door for further investigation into the role of memory, speed of learning, and appropriate institutions that could facilitate the search along a path to stability.

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